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Spectral characteristics of non-stationary random processes: Theory and applications to linear structural models

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Abstract

The spectral characteristics are important quantities in describing random processes. Proper definitions of these quantities are available for real-valued stationary and non-stationary processes. In this paper, the well-established definitions of spectral characteristics for real-valued stationary and non-stationary processes are extended to general complex-valued non-stationary random processes. This extension allows to derive the exact solution in closed-form for the classical problem of computing the time-variant central frequency and bandwidth parameter of the response processes of single-degree-of-freedom (SDOF) and both classically and non-classically damped multi-degree-of-freedom (MDOF) linear elastic systems subjected to white noise excitation from at rest initial conditions. These new exact closed-form solutions are also used to gain deeper insight into the time-variant and stationary behavior of the central frequency and bandwidth parameter of these linear response processes.

Keywords: Stationary and non-stationary stochastic processes; Spectral moments; Non-geometric spectral characteristics; Classically and non-classically damped MDOF systems

1. Introduction

The probabilistic study of the dynamic behavior of structural and mechanical systems requires the characterization of the random processes describing the input excitation and the structural response. This characterization is usually very complex for realistic input processes and structural systems, when non-stationary and non-Gaussian processes are involved.

A very common and powerful methodology for characterizing and describing a random process is spectral analysis, which studies random processes in the frequency domain. In particular, the use of power spectral density (PSD) functions [1] is customary in describing stationary random processes. Definition of functions describing the spectral properties of non-stationary random processes is less simple and not unique. In fact, several non-stationary spectra have been defined in the literature [2,3], with different application fields. In addition, direct extension of the definition of spectral characteristics, such as the spectral moments, from stationary to non-stationary processes leads to difficulties in the interpretation and application of these spectral characteristics [4].

Among existing definitions of non-stationary spectra, the most widely used is probably Priestley's evolutionary power spectral density (EPSD) [1]. Based on this EPSD, the so-called "non-geometric" spectral characteristics (NGSCs) have been defined for real-valued non-stationary processes [5,6]. The NGSCs have been proved appropriate for describing non-stationary processes [7] and can be effectively employed in structural reliability applications, such as the computation of the time-variant probability that a random process outcrosses a given limit-state threshold.

In this paper, the definition of NGSCs is extended to general complex-valued non-stationary random processes. These newly defined quantities provide information consistent with that provided by their counterparts for real-valued stationary and non-stationary processes. These NGSCs are used in this study to solve exactly and in closed-form the classical problem of computing the time-variant central frequency and

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bandwidth parameter of the response processes of single-degree-of-freedom (SDOF) and both classically and non-classically damped multi-degree-of-freedom (MDOF) linear elastic systems subjected to white noise excitation from at rest initial conditions. In addition, the NGSCs of complex-valued processes are useful in problems which require the use of complex modal analysis, such as random vibrations of non-classically damped MDOF linear structures, and in structural reliability applications [8], for which the existing definitions of spectral characteristics were specifically developed.

For the sake of simplicity and without loss of generality, all random processes considered in this study are zero-mean processes. An important implication is that the auto- and cross-covariance functions of these random processes coincide with their auto- and cross-correlation functions, respectively.

2. Central frequency and bandwidth parameters for real-valued stochastic processes

A real-valued stationary process $X_S(t)$ has the following spectral decomposition:

$$X_S(t) = \int_{-\infty}^{\infty} \mathrm{e}^{j\omega t} \mathrm{d}Z(\omega) \tag{1}$$

in which t = time, ω = frequency parameter, $j = \sqrt{-1}$, and $\mathrm{d}Z(\omega)$ = zero-mean orthogonal increment process defined so that $E[\mathrm{d}Z^*(\omega_1)\mathrm{d}Z(\omega_2)] = \Phi(\omega_1)\delta(\omega_1 - \omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2$ where $E[...]$ = mathematical expectation, $\Phi(\omega)$ = PSD function of the stationary process $X_S(t)$, $\delta(\ldots)$ = Dirac delta function and the superscript $(\ldots)^*$ denotes the complex-conjugate operator. For the stationary process considered, $X_S(t)$, the geometric spectral moments λ_n of order n $(n = 0, 1, \ldots)$ are defined as [13]

$$\lambda_n = \int_{-\infty}^{\infty} |\omega|^n \Phi(\omega)\mathrm{d}\omega = 2\int_0^{\infty} \omega^n \Phi(\omega)\mathrm{d}\omega \tag{2}$$

where $|\ldots|$ = absolute value of a real-valued variable (or modulus of a complex-valued variable). The geometric spectral moments are utilized in random vibration problems to compute several meaningful quantities, such as

(1) The variance of the ith time-derivative of the process $X_S(t)$, $X_S^{(i)}(t)$ (provided that this ith time-derivative process exists in the mean-square sense): $\sigma^2_{X_S^{(i)}} = \lambda_{2i}$ $(i = 0, 1, \ldots)$.

(2) The central frequency parameter ω_c of the process $X_S(t)$:

$$\omega_c = \frac{\lambda_1}{\lambda_0}. \tag{3}$$

(3) The bandwidth parameter q of the process $X_S(t)$:

$$q = \left(1 - \frac{\lambda_1^2}{\lambda_0\lambda_2}\right)^{\frac{1}{2}}. \tag{4}$$

Similarly to the stationary case, a real-valued non-stationary (RVNS) process $X(t)$ can be expressed in the general form of a Fourier–Stieltjes integral as [1]

$$X(t) = \int_{-\infty}^{\infty} A_X(\omega, t)\mathrm{e}^{j\omega t}\mathrm{d}Z(\omega) \tag{5}$$

where $A_X(\omega, t)$ = complex-valued deterministic time-frequency modulating function defined such that

$$A_X(-\omega, t) = A_X^*(\omega, t). \tag{6}$$

An embedded stationary process $X_S(t)$, with PSD function $\Phi(\omega)$, is associated to the RVNS process $X(t)$. The process $X(t)$ has the following EPSD function:

$$\Phi_{XX}(\omega, t) = A_X^*(\omega, t) \cdot \Phi(\omega) \cdot A_X(\omega, t). \tag{7}$$

From Eqs. (6) and (7), it is seen that the EPSD of a RVNS process is a symmetric function of the frequency parameter ω.

The definition of the geometric spectral moments in Eq. (2) can be mathematically extended to the non-stationary case as

$$\lambda_n(t) = \int_{-\infty}^{\infty} |\omega|^n \Phi_{XX}(\omega, t)\mathrm{d}\omega = 2\int_0^{\infty} \omega^n \Phi_{XX}(\omega, t)\mathrm{d}\omega. \tag{8}$$

Using these spectral moments, Corotis et al. [4] extended consistently the definitions of the central frequency, $\omega_c(t)$, and bandwidth parameter, $q(t)$, to RVNS processes. The geometric spectral moments defined in Eq. (8) suffer two severe drawbacks in characterizing non-stationary stochastic processes [6], namely

(1) The variance of the ith time-derivative of the process $X(t)$ for $i > 0$ is not equal to the $2i$th spectral moment.

(2) Even when the variance of the ith time-derivative of the process is finite, the $2i$th non-stationary geometric spectral moment can be divergent, in which case the consistent definition of central frequency and bandwidth parameter in terms of geometric spectral moments cannot be computed.

More recently, Di Paola [5] and Michaelov et al. [6,7] introduced a proper definition of spectral characteristics to be used in computing the central frequency and bandwidth parameter for a RVNS process $X(t)$ defined by Eq. (5) through (7). For such a process, the so-called "non-geometric" spectral characteristics (NGSCs) $c_{ik}(t)$ are defined as

$$c_{ik}(t) = 2(-1)^k j^{i+k} \int_0^{\infty} \Phi_{X^{(i)}X^{(k)}}(\omega, t)\mathrm{d}\omega, \quad i, k = 0, 1, \ldots \tag{9}$$

where $\Phi_{X^{(i)}X^{(k)}}(\omega, t)$ is the evolutionary cross-PSD function of the time-derivatives of order i and k of the process $X(t)$, i.e.,

$$\Phi_{X^{(i)}X^{(k)}}(\omega, t) = A^*_{X^{(i)}}(\omega, t) \cdot \Phi(\omega) \cdot A_{X^{(k)}}(\omega, t), \quad i, k = 0, 1, \ldots \tag{10}$$

in which $X^{(m)}(t) = \mathrm{d}^m X(t)/\mathrm{d}t^m$ $(m = i, k)$, provided that $X^{(m)}(t)$ exists in the mean-square sense, and the modulating function $A_{X^{(m)}}(\omega, t)$ is obtained recursively [6]. This definition of NGSCs is equivalent to the one derived by Di Paola [5] from the Rice envelope process. These NGSCs can also be derived from the auto-correlation function of the complex-valued pre-envelope process [9,10] of process $X(t)$, as shown by Krenk et al. [11] and Krenk and Madsen [12].

At this point, it is convenient to define the process $Y(t)$ as the modulation (with modulating function $A_X(\omega,t)$) of the stationary process $Y_S(t)$ defined as the Hilbert transform of the embedded stationary process $X_S(t)$ [9,10], i.e.,

$$Y(t) = -j\int_{-\infty}^{\infty} \mathrm{sign}(\omega) A_X(\omega,t) \mathrm{e}^{j\omega t} \mathrm{d}Z(\omega). \tag{11}$$

Using the NGSCs in Eq. (9) and then Eq. (11), the time-variant central frequency $\omega_c(t)$ and bandwidth parameter $q(t)$ are defined as [6]

$$\omega_c(t) = \frac{(\mathrm{Re}[c_{01}(t)])}{c_{00}(t)} = \frac{\sigma_{X\dot{Y}}(t)}{\sigma_X^2(t)} \tag{12}$$

$$q(t) = \left(1 - \frac{(\mathrm{Re}[c_{01}(t)])^2}{c_{00}(t)c_{11}(t)}\right)^{\frac{1}{2}} = \left(1 - \frac{\sigma_{X\dot{Y}}^2(t)}{\sigma_X^2(t)\sigma_{\dot{X}}^2(t)}\right)^{\frac{1}{2}} \tag{13}$$

where Re[...] = real part of the quantity in square brackets and the NGSC $c_{01}(t)$ is expressed as

$$\begin{aligned} c_{01}(t) &= c_{10} * (t) = -2j\int_0^{\infty} \Phi_{X\dot{X}}(\omega,t)\mathrm{d}\omega \\ &= \sigma_{X\dot{Y}}(t) - j\sigma_{X\dot{X}}(t) \end{aligned} \tag{14}$$

where $\sigma_{X\dot{Y}}(t)$ = cross-covariance of $X(t)$ and $\dot{Y}(t)$, and $\sigma_{X\dot{X}}(t)$ = cross-covariance of $X(t)$ and $\dot{X}(t)$. Notice that, in the case of a stationary process, Eqs. (12) and (13) reduce to Eqs. (3) and (4), respectively.

The time-variant central frequency and bandwidth parameter are useful in describing the time-variant spectral properties of a RVNS process $X(t)$. The central frequency $\omega_c(t)$ provides the characteristic/predominant frequency of the process at each instant of time. The bandwidth parameter $q(t)$ provides information on the spectral bandwidth of the process at each instant of time. Notice that a non-stationary process can behave as a narrowband and a broadband process at different instants of time. In addition, the bandwidth parameter $q(t)$ plays an important role in time-variant reliability analysis, since it is an essential ingredient of analytical approximations [8,13] to the time-variant failure probability for the first-passage reliability problem [14–16].

3. Spectral characteristics of complex-valued non-stationary stochastic processes

The definition in Eq. (5) can be mathematically extended to complex-valued non-stationary (CVNS) processes with a general complex-valued deterministic time-frequency modulating function $A_X(\omega,t)$. In this case, Eq. (6) does not hold in general and the EPSD is not a symmetric function of the frequency parameter ω. Notice that this extension applies to complex-valued processes which are more general than the pre-envelope process introduced by Arens [9] and Dugundji [10].

In this paper, an extension of the definition of NGSCs to CVNS random processes is proposed and presented. For CVNS processes, the real and imaginary parts of the evolutionary cross-PSD function $\Phi_{X^{(i)}X^{(k)}}(\omega,t)$ are not symmetric and anti-symmetric functions, respectively, of the frequency parameter ω. Our interest is limited to CVNS processes with a real-valued embedded stationary process $X_S(t)$ as defined by Eq. (1).

For each CVNS process $X(t)$, two sets of NGSCs are defined as follows

$$\begin{cases} c_{ik,XX}(t) = \displaystyle\int_{-\infty}^{\infty} \Phi_{X^{(i)}X^{(k)}}(\omega,t)\mathrm{d}\omega = \sigma_{X^{(i)}X^{(k)}}(t) \\ c_{ik,XY}(t) = \displaystyle\int_{-\infty}^{\infty} \Phi_{X^{(i)}Y^{(k)}}(\omega,t)\mathrm{d}\omega = \sigma_{X^{(i)}Y^{(k)}}(t) \\ i,k = 0,1,\ldots \end{cases} \tag{15}$$

where $\sigma_{X^{(i)}X^{(K)}}(t)$ = cross-covariance of random processes $X^{(i)}(t)$ and $X^{(k)}(t)$, and $\sigma_{X^{(i)}Y^{(K)}}(t)$ = cross-covariance of random processes $X^{(i)}(t)$ and $Y^{(k)}(t) = \mathrm{d}^kY(t)/\mathrm{d}t^k$. The process $Y(t)$ is defined by Eq. (11), and the evolutionary cross-PSD functions $\Phi_{X^{(i)}W^{(k)}}(\omega,t)$ ($W = X, Y$, and $i,k = 0,1,\ldots$) are given by

$$\begin{aligned} &\Phi_{X^{(i)}W^{(k)}}(\omega,t) = A^*_{X^{(i)}}(\omega,t)\cdot\Phi(\omega)\cdot A_{W^{(k)}}(\omega,t); \\ &\quad W = X, Y;\ i = 0,1,\ldots \end{aligned} \tag{16}$$

where [17]

$$\begin{aligned} &A_{W^{(i)}}(\omega,t) = \mathrm{e}^{-j\omega t}\frac{\partial^i}{\partial t^i}[A_W(\omega,t)\cdot\mathrm{e}^{j\omega t}]; \\ &\quad W = X, Y;\ i = 0,1,\ldots. \end{aligned} \tag{17}$$

Again, it is assumed that the time-derivative processes in Eq. (15) exist in the mean-square sense. In the particular case when $i = k = n$, the cross-covariance in Eq. $(15)_1$ reduces to the variance of the nth time-derivative of the process $X(t)$, i.e., $\sigma_{X^{(n)}X^{(n)}}(t) = \sigma^2_{X^{(n)}}(t)$. The four NGSCs $c_{00,XX}(t)$, $c_{11,XX}(t)$, $c_{01,XX}(t)$ and $c_{01,XY}(t)$ are particularly relevant to random vibration theory and time-variant reliability applications. In fact, $c_{00,XX}(t)$ and $c_{11,XX}(t)$ represent the variance of the process and its first time-derivative (i.e., $\sigma_X^2(t)$ and $\sigma_{\dot{X}}^2(t)$), respectively, $c_{01,XX}(t)$ denotes the cross-covariance of the process and its first time-derivative (i.e., $\sigma_{X\dot{X}}(t)$), and $c_{01,XY}(t)$ represents the cross-covariance of the process $X(t)$ and the first time-derivative of the process $Y(t)$ (i.e., $\sigma_{X\dot{Y}}(t)$).

Notice that for RVNS processes, the definitions in Eq. (15) for $c_{00,XX}(t)$, $c_{11,XX}(t)$, $c_{01,XX}(t)$and $c_{01,XY}(t)$ are equivalent to the definitions in Eq. (9) for $i,k = 0,1$, since

$$\begin{cases} c_{00}(t) = 2(-1)^0 j^0 \displaystyle\int_0^{\infty} \Phi_{XX}(\omega,t)\mathrm{d}\omega \\ \quad = \displaystyle\int_{-\infty}^{\infty} \Phi_{XX}(\omega,t)\mathrm{d}\omega = c_{00,XX}(t) \\ c_{11}(t) = 2(-1)^1 j^2 \displaystyle\int_0^{\infty} \Phi_{\dot{X}\dot{X}}(\omega,t)\mathrm{d}\omega \\ \quad = \displaystyle\int_{-\infty}^{\infty} \Phi_{\dot{X}\dot{X}}(\omega,t)\mathrm{d}\omega = c_{11,XX}(t) \\ c_{01}(t) = 2(-1)^1 j^1 \displaystyle\int_0^{\infty} \Phi_{X\dot{X}}(\omega,t)\mathrm{d}\omega \\ \quad = c_{01,XY}(t) - jc_{01,XX}(t). \end{cases} \tag{18}$$

The NGSCs $c_{00,XX}(t)$, $c_{11,XX}(t)$ and $c_{01,XY}(t)$ are used in the definition of the time-variant central frequency, $\omega_c(t)$, and bandwidth parameter, $q(t)$, of the CVNS process $X(t)$ as

$$\omega_c(t) = \frac{c_{01,XY}(t)}{c_{00,XX}(t)} = \frac{\sigma_{X\dot{Y}}(t)}{\sigma_X^2(t)} \tag{19}$$

$$q(t) = \left(1 - \frac{[c_{01,XY}(t)]^2}{\sigma_X^2(t)\sigma_{\dot{X}}^2(t)}\right)^{\frac{1}{2}} = \left(1 - \frac{\sigma_{X\dot{Y}}^2(t)}{\sigma_X^2(t)\sigma_{\dot{X}}^2(t)}\right)^{\frac{1}{2}}. \tag{20}$$

In the case of RVNS processes, the two definitions in Eqs. (19) and (20) reduce to the ones given in Eqs. (12) and (13), respectively. However, for CVNS processes, the complex-valued central frequency and bandwidth parameter defined in Eqs. (19) and (20) loose the simple physical interpretation available for RVNS processes.

4. Spectral characteristics of the stochastic response of SDOF and MDOF linear systems subjected to non-stationary input excitation

4.1. Complex modal analysis

A state-space formulation of the equations of motion for a linear MDOF system is useful to describe the response of both classically and non-classically damped systems [18]. The general (second-order) equations of motion for an n-degree-of-freedom linear system are, in matrix form,

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) = \mathbf{P}F(t) \tag{21}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K} = n \times n$ time-invariant mass, damping and stiffness matrices, respectively; $\mathbf{U}(t)$, $\dot{\mathbf{U}}(t)$, and $\ddot{\mathbf{U}}(t) =$ length-n vectors of nodal displacements, velocities and accelerations, respectively; $\mathbf{P} =$ length-n load distribution vector, and $F(t) =$ scalar function describing the time-history of the external loading which, in the case of random excitation, is modeled as a random process. Defining the following length-$2n$ state vector

$$\mathbf{Z}(t) = \begin{bmatrix} \mathbf{U}(t) \\ \dot{\mathbf{U}}(t) \end{bmatrix}_{(2n\times 1)}, \tag{22}$$

the matrix equation of motion (21) can be recast into the following first-order matrix equation

$$\dot{\mathbf{Z}}(t) = \mathbf{G}\mathbf{Z}(t) + \tilde{\mathbf{P}}F(t) \tag{23}$$

where

$$\mathbf{G} = \begin{bmatrix} \mathbf{0}_{(n\times n)} & \mathbf{I}_{(n\times n)} \\ (-\mathbf{M}^{-1}\mathbf{K}) & (-\mathbf{M}^{-1}\mathbf{C}) \end{bmatrix}_{(2n\times 2n)} \tag{24}$$

$$\tilde{\mathbf{P}} = \begin{bmatrix} \mathbf{0}_{(n\times 1)} \\ \mathbf{M}^{-1}\mathbf{P} \end{bmatrix}_{(2n\times 1)}. \tag{25}$$

The subscripts in Eqs. (22), (24) and (25) indicate the dimensions of the vectors and matrices to which they are attached. The complex modal matrix, $\mathbf{T}$, is formed from the complex eigenmodes of matrix $\mathbf{G}$ and can be used as an appropriate transformation matrix to decouple the first-order matrix equation (23) and introduce the transformed state vector $\mathbf{V}(t)$ of complex modal coordinates as

$$\mathbf{Z}(t) = \mathbf{T}\mathbf{V}(t). \tag{26}$$

Substituting Eq. (26) into Eq. (23), considering that $\mathbf{T}^{-1}\mathbf{G}\mathbf{T} = \mathbf{D}$ [18], where $\mathbf{D}$ is a diagonal matrix containing the $2n$ complex eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$, of the system matrix $\mathbf{G}$, and $\mathbf{T}^{-1}\tilde{\mathbf{P}} = [\Gamma_1, \ldots, \Gamma_{2n}]^{\mathrm{T}}$ where Γ_i is the ith modal participation factor (complex-valued), the normalized complex modal equations are obtained as

$$\dot{S}_i(t) = \lambda_i S_i(t) + F(t), \quad i = 1, 2, \ldots, 2n \tag{27}$$

where the normalized complex modal responses $S_i(t)$ $(i = 1, 2, \ldots, 2n)$ are defined as

$$S_i(t) = \frac{1}{\Gamma_i} V_i(t), \quad i = 1, 2, \ldots, 2n. \tag{28}$$

The impulse response function for the ith mode, $h_i(t)$, defined as the solution of Eq. (27) when $F(t) = \delta(t)$ and for at rest initial conditions at time $t = 0^-$ (i.e., $S_{(i)}(0^-) = 0$), is simply given by $h_{(i)}(t) = \mathrm{e}^{\lambda_i t}$ $(t > 0)$. Assuming that the system is initially at rest, the solution of Eq. (27) can be expressed by the following Duhamel integral:

$$S_i(t) = \int_0^t \mathrm{e}^{\lambda_i(t-\tau)} F(\tau)\mathrm{d}\tau, \quad i = 1, 2, \ldots, 2n. \tag{29}$$

It is worth mentioning that the normalized complex modal responses $S_i(t), i = 1, 2, \ldots, 2n$, are complex conjugate by pairs and in this study are ordered so that $S_i(t) = S_{n+i}^*(t)$. In the case of a non-stationary loading process, the loading function $F(t)$ can be expressed in general as (see Eq. (5)).

$$F(t) = \int_{-\infty}^{\infty} A_F(\omega, t)\mathrm{e}^{j\omega t}\mathrm{d}Z(\omega). \tag{30}$$

It can be shown that the normalized complex modal responses are given by

$$S_i(t) = \int_{-\infty}^{\infty} A_{S_i}(\omega, t)\mathrm{e}^{j\omega t}\mathrm{d}Z(\omega), \quad i = 1, 2, \ldots, 2n \tag{31}$$

where

$$A_{S_i}(\omega, t) = \int_0^t \{\mathrm{e}^{\lambda_i(t-\tau)} A_F(\omega, \tau) \cdot \mathrm{e}^{j\omega(\tau-t)}\}\mathrm{d}\tau, \quad i = 1, 2, \ldots, 2n. \tag{32}$$

Combining Eqs. (26) and (28) yields

$$\mathbf{Z}(t) = \mathbf{T}\mathbf{V}(t) = \mathbf{T}\boldsymbol{\Gamma}\mathbf{S}(t) = \tilde{\mathbf{T}}\mathbf{S}(t) \tag{33}$$

in which $\boldsymbol{\Gamma} =$ diagonal matrix containing the $2n$ modal participation factors Γ_i, $\tilde{\mathbf{T}} = \mathbf{T}\boldsymbol{\Gamma} =$ effective modal participation matrix and $\mathbf{S} = [S_1(t), S_2(t), \ldots, S_{2n}(t)]^{\mathrm{T}} =$ normalized complex modal response vector.

4.2. Non-geometric spectral characteristics of response processes of linear MDOF systems using complex modal analysis

The state-space formulation of the equations of motion is also advantageous for the computation of the NGSCs of response processes of both classically and non-classically damped linear MDOF systems. If only Gaussian input processes are considered, only few spectral characteristics are needed to fully describe the response processes of linear elastic MDOF systems, since the response processes are also Gaussian. In particular, if $U_i(t)$ denotes the ith DOF displacement response process of a linear elastic MDOF system subjected to Gaussian excitation, the only spectral characteristics required, e.g., for reliability applications, are

$$\begin{cases} c_{00,U_iU_i}(t) = \sigma^2_{U_i}(t), \\ c_{11,U_iU_i}(t) = \sigma^2_{\dot{U}_i}(t), \\ c_{01,U_iU_i}(t) = \sigma_{U_i\dot{U}_i}(t), \\ c_{01,U_i\Upsilon_i}(t) = \sigma_{U_i\dot{\Upsilon}_i}(t), \end{cases} \quad i = 1, 2, \ldots, n \tag{34}$$

where $\dot{\Upsilon}_i(t)$ denotes the first time-derivative of the process $\Upsilon_i(t)$ defined as

$$\Upsilon_i(t) = -j\int_{-\infty}^{\infty} \operatorname{sign}(\omega) A_{U_i}(\omega, t) \mathrm{e}^{j\omega t} \mathrm{d}Z(\omega), \quad i = 1, 2, \ldots, n \tag{35}$$

and $A_{U_i}(\omega, t)$ denotes the time-frequency modulating function of process $U_i(t)$. The process $\Upsilon_i(t)$ is the modulation (with the same modulating function $A_{U_i}(\omega, t)$ as process $U_i(t)$) of the Hilbert transform of the stationary process embedded in the process $U_i(t)$ (see Eq. (11) and [9,10]).

Similarly to the response processes (see Eq. (22)), the following auxiliary state vector process can be defined

$$\boldsymbol{\Xi}(t) = \begin{bmatrix} \boldsymbol{\Upsilon}(t) \\ \dot{\boldsymbol{\Upsilon}}(t) \end{bmatrix}_{(2n\times 1)}. \tag{36}$$

Using complex modal decomposition, the cross-covariance matrices of the response processes and the auxiliary processes can be computed as

$$\begin{aligned} E[\mathbf{Z}(t)\mathbf{Z}^{\mathrm{T}}(t)] &= E\begin{bmatrix} \mathbf{U}(t)\mathbf{U}^{\mathrm{T}}(t) & \mathbf{U}(t)\dot{\mathbf{U}}^{\mathrm{T}}(t) \\ \dot{\mathbf{U}}(t)\mathbf{U}^{\mathrm{T}}(t) & \dot{\mathbf{U}}(t)\dot{\mathbf{U}}^{\mathrm{T}}(t) \end{bmatrix} \\ &= \tilde{\mathbf{T}}^* E[\mathbf{S}^*(t)\mathbf{S}^{\mathrm{T}}(t)]\tilde{\mathbf{T}}^{\mathrm{T}} \end{aligned} \tag{37}$$

$$\begin{aligned} E[\mathbf{Z}(t)\boldsymbol{\Xi}^{\mathrm{T}}(t)] &= E\begin{bmatrix} \mathbf{U}(t)\boldsymbol{\Upsilon}^{\mathrm{T}}(t) & \mathbf{U}(t)\dot{\boldsymbol{\Upsilon}}^{\mathrm{T}}(t) \\ \dot{\mathbf{U}}(t)\boldsymbol{\Upsilon}^{\mathrm{T}}(t) & \dot{\mathbf{U}}(t)\dot{\boldsymbol{\Upsilon}}^{\mathrm{T}}(t) \end{bmatrix} \\ &= \tilde{\mathbf{T}}^* E[\mathbf{S}^*(t)\boldsymbol{\Sigma}^{\mathrm{T}}(t)]\tilde{\mathbf{T}}^{\mathrm{T}} \end{aligned} \tag{38}$$

where the components of the vector process $\boldsymbol{\Sigma} = [\Sigma_1(t), \Sigma_2(t), \ldots, \Sigma_{2n}(t)]^{\mathrm{T}}$ are defined as

$$\Sigma_i(t) = -j\int_{-\infty}^{\infty} \operatorname{sign}(\omega) A_{S_i}(\omega, t) \mathrm{e}^{j\omega t} \mathrm{d}Z(\omega), \quad i = 1, 2, \ldots, 2n. \tag{39}$$

Eqs. (37) and (38) show that all quantities in Eq. (34) can be computed from the following spectral characteristics of complex-valued non-stationary processes

$$\begin{cases} E[S_i^*(t)S_m(t)] = \sigma_{S_iS_m}(t) \\ E[S_i^*(t)\Sigma_m(t)] = \sigma_{S_i\Sigma_m}(t) \end{cases} \quad i, m = 1, 2, \ldots, 2n. \tag{40}$$

Notice also that knowledge of the spectral characteristics in Eq. (40) allows computation of the zeroth- to second-order spectral characteristics of the components of any vector response quantity $\mathbf{Q}(t)$ linearly related to the displacement response vector $\mathbf{U}(t)$, i.e., $\mathbf{Q}(t) = \mathbf{B}\mathbf{U}(t)$, where $\mathbf{B}$ = constant matrix.

4.3. Response statistics of MDOF linear systems subjected to modulated Gaussian white noise

Time-modulated Gaussian white noises constitute an important class of non-stationary dynamic load processes. The expression given in Eq. (30) describing a general non-stationary loading process reduces to

$$F(t) = A_F(t) \cdot W(t) \tag{41}$$

where the time-modulating function $A_F(t)$ is frequency-independent and the white noise process $W(t)$ has a constant PSD equal to ϕ_0.

In the following, closed-form solutions are derived for the case of the modulating function equal to the unit-step function, i.e., $A_F(t) = H(t)$. Notice that even for this very simple modulating function and for a SDOF linear oscillator, to date and to the best of the authors' knowledge, no closed-form solution is available for the first-order NGSC $c_{01,U\Upsilon}(t)$ ($\mathrm{Re}[c_{01}(t)]$ in the notation adopted by Michaelov et al. [6, 7]) required for computing the time-variant central frequency and bandwidth parameter of the displacement response process $U(t)$. In the following, the presented extension of NGSCs to complex-valued non-stationary stochastic processes is employed to derive the closed-form solution for $c_{01,U\Upsilon}(t)$.

In the case of the unit-step modulating function, Eq. (32) becomes

$$\begin{aligned} A_{S_i}(\omega, t) &= \mathrm{e}^{(\lambda_i - j\omega)t}\int_0^t \{H(\tau)\cdot \mathrm{e}^{-(\lambda_i - j\omega)\tau}\}\mathrm{d}\tau \\ &= \frac{\mathrm{e}^{(\lambda_i - j\omega)t} - 1}{\lambda_i - j\omega}, \quad i = 1, 2, \ldots, 2n. \end{aligned} \tag{42}$$

The spectral characteristics in Eq. $(40)_1$ can be computed using Cauchy's residue theorem as [19]

$$\sigma_{S_iS_m}(t) = \frac{2\pi\phi_0}{\lambda_i^* + \lambda_m}[\mathrm{e}^{(\lambda_i^* + \lambda_m)t} - 1], \quad i, m = 1, 2, \ldots, 2n. \tag{43}$$

Krenk and Madsen [12] and Madsen and Krenk [20] applied the same approach (integration using Cauchy's residue theorem) to the real-valued (second-order) modal responses to derive the closed-form solutions for the auto- and cross-correlation functions of the response processes of classically damped MDOF linear systems subjected to white noise excitations modulated by rational time-modulating functions. After extensive algebraic manipulations [19], the spectral characteristics in Eq. $(40)_2$ are obtained as

$$\sigma_{S_i \Sigma_m}(t) = \frac{2\phi_0}{\lambda_i^* + \lambda_m} \times [E_1(-\lambda_i^* t) + \log(-\lambda_i^*) - E_1(-\lambda_m t) - \log(-\lambda_m)] + \frac{2\phi_0}{\lambda_i^* + \lambda_m} \mathrm{e}^{(\lambda_i^* + \lambda_m)t} \times [E_1(\lambda_i^* t) + \log(\lambda_i^*) - E_1(\lambda_m t) - \log(\lambda_m)], \quad i, m = 1, 2, \ldots, 2n \tag{44}$$

in which $E_1(x)$ denotes the integral exponential function defined as [21]

$$E_1(x) = \int_x^\infty \frac{\mathrm{e}^{-u}}{u} \mathrm{d}u, \quad |\arg(x)| < \pi \tag{45}$$

where $\arg(\ldots)$ = complex argument function.

The introduction of the spectral characteristics of the complex modal response processes $S_i(t)$ $(i = 1, \ldots, 2n)$ has the following important advantages:

(1) Closed-form integration for variances and cross-covariances of displacement and velocity response processes for linear elastic MDOF systems can be performed using Cauchy's residue theorem provided that the time-frequency modulating functions are rational functions, which is not a severe restriction.
(2) The time-frequency modulating functions of response processes are obtained by integrating (in closed-form or numerically) Eq. (32), in which the time-frequency modulating function of the loading process, $A_F(\omega, t)$, is multiplied by the impulse response function of a first-order differential equation (i.e., $h(t) = \mathrm{e}^{\lambda t}$, $t > 0$, λ = complex-valued constant). In general, this integration is much simpler than its counterpart for real-valued (second-order) modal response processes, in which the time-modulating function of the loading process is multiplied by the impulse response function of a second-order differential equation (i.e., $h(t) = \frac{\mathrm{e}^{-\mathrm{Re}[\lambda] \cdot t}}{\mathrm{Im}[\lambda]} \sin(\mathrm{Im}[\lambda] \cdot t)$, $t > 0$, where $\mathrm{Im}[\ldots]$ = imaginary part of the quantity in the square brackets).
(3) The use of complex modal decomposition allows computation of the spectral characteristics of response quantities of linear MDOF systems that are non-classically damped.
(4) The presented extension of NGSCs to complex-valued non-stationary stochastic processes enables the derivation of the exact solution in closed-form for the first-order NGSC, $c_{01,U_i \Upsilon_i}(t)$ (see Eq. $(34)_4$), of response processes of linear SDOF and MDOF systems subjected to white noise excitation modulated by the unit-step function. This closed-form solution cannot be obtained using real-valued responses of second-order modes.

5. Application examples

5.1. Linear elastic SDOF systems

The first application example consists of a set of linear elastic SDOF systems subjected to a Gaussian white noise time-modulated by the unit-step function (i.e., from at rest initial conditions). In this case, the complex modal matrix $\mathbf{T}$ is given by

$$\mathbf{T} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \tag{46}$$

in which

$$\lambda_{1,2} = -\xi\omega_0 \pm j\omega_d \tag{47}$$

where ξ = viscous damping ratio, ω_0 = natural circular frequency, and $\omega_d = \omega_0\sqrt{1-\xi^2}$ = damped circular frequency of the system. It is assumed that $0 < \xi < 1$, which is usually the case for structural systems.

From Eqs. (33), (43) and (46), the well-known closed-form solutions for the variances of the displacement and velocity response processes and the cross-covariance between the displacement and the velocity response processes [19,22] are readily obtained. After some algebraic manipulations [19], the first-order NGSC $\sigma_{U\dot{\Upsilon}}(t)$ is found as

$$\sigma_{U\dot{\Upsilon}}(t) = \frac{j\phi_0}{2\xi\omega_0\omega_d} \times \left[E_1(-\lambda_1 t) - E_1(-\lambda_2 t) - 2j \cdot \mathrm{arctg}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) \right] + \frac{j\phi_0}{2\xi\omega_0\omega_d} \mathrm{e}^{-(2\xi\omega_0)t} \times \left\{ E_1(\lambda_1 t) - E_1(\lambda_2 t) + 2j \left[\pi - \mathrm{arctg}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) \right] \right\}. \tag{48}$$

It is worth noting that Eq. (48) can be directly employed for computing the corresponding first-order NGSCs of the response processes of linear MDOF systems that are classically damped, by using real-valued (second-order) mode superposition and thus avoiding complex modal analysis, which is computationally more expensive and less commonly used. From Eq. (48), the stationary value of the spectral characteristic $\sigma_{U\dot{\Upsilon}}(t)$ is readily obtained as

$$\sigma_{U\dot{\Upsilon},\infty} = \frac{\phi_0}{\xi\omega_0\omega_d} \mathrm{arctg}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right). \tag{49}$$

The result provided in Eq. (48) is the exact closed-form solution for the NGSC $\sigma_{U\dot{\Upsilon}}(t)$. To date and to the best of the authors' knowledge, $\sigma_{U\dot{\Upsilon}}(t)$ could only be obtained by

(1) evaluating numerically the following exact expression in integral form [7]

$$\sigma_{U\dot{\Upsilon}}(t) = \frac{\phi_0}{\xi\omega_0\omega_d} \left[\int_0^t \mathrm{e}^{-\xi\omega_0 u} \frac{\sin(\omega_d u)}{u} \mathrm{d}u - \mathrm{e}^{-2\xi\omega_0 t} \int_0^t \mathrm{e}^{\xi\omega_0 u} \frac{\sin(\omega_d u)}{u} \mathrm{d}u \right] = \mathrm{Re}[c_{01}(t)]. \tag{50}$$

(2) time-differentiating the time-variant cross-correlation function of the response process $U(t)$ and its Hilbert transform

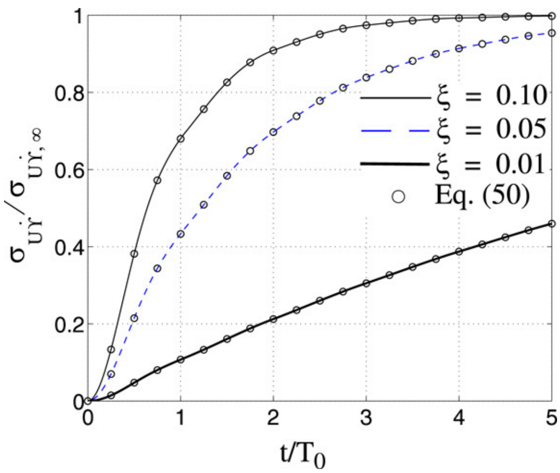

Fig. 1. Comparison of analytical (Eq. (48)) and numerical (Eq. (50)) solutions for the normalized first-order NGSC, $\sigma_{U\dot{\Upsilon}}(t)/\sigma_{U\dot{\Upsilon},\infty}$, of the displacement responses of linear SDOF systems with damping ratios $\xi = 0.01, 0.05$ and 0.10.

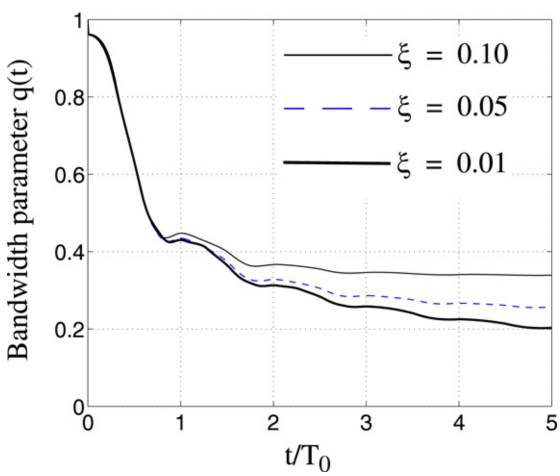

Fig. 2. Time-variant bandwidth parameter, $q(t)$, of the displacement responses of linear SDOF systems with damping ratios $\xi = 0.01, 0.05$ and 0.10.

$\Upsilon(t)$ obtained from the auto-correlation function of the complex-valued pre-envelope process of process $U(t)$ [11, 12], which also requires the numerical evaluation of an integral term.

Fig. 1 shows the first-order NGSC $c_{01,U\Upsilon}(t) = \sigma_{U\dot{\Upsilon}}(t)$ (Eq. (48)) normalized by its stationary value $\sigma_{U\dot{\Upsilon},\infty}$ (Eq. (49)) for SDOF systems with three different damping ratios (i.e., $\xi = 0.01, 0.05, 0.10$). For comparison purposes, Fig. 1 also provides the normalized first-order NGSC, $\sigma_{U\dot{\Upsilon}}(t)/\sigma_{U\dot{\Upsilon},\infty}$, with the numerator evaluated numerically through Eq. (50). The normalized first-order NGSC $\sigma_{U\dot{\Upsilon}}(t)/\sigma_{U\dot{\Upsilon},\infty}$, is a function of the damping ratio and the time normalized by the natural period T_0 of the SDOF system considered. As expected, stationarity is reached after a larger number of response cycles (periods) for decreasing value of the damping ratio ξ.

Fig. 2 plots the bandwidth parameter $q(t)$ of the displacement responses of SDOF systems with $\xi = 0.01, 0.05, 0.10$. From the results in Fig. 2, it is observed that:

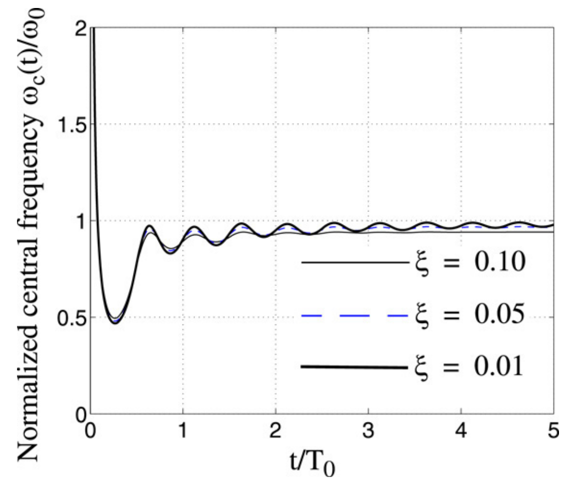

Fig. 3. Time-variant normalized central frequency of the displacement responses of linear SDOF systems with damping ratios $\xi = 0.01, 0.05$ and 0.10.

(1) The value of $q(t)$ at $t = 0$ s is always equal to 0.961. This result implies that, at the start of the motion of the system, the SDOF system response is broadband.
(2) The value of $q(t)$ decreases in time until it reaches a stationary value, i.e., the SDOF system response changes from a broadband transient to a narrowband stationary process.
(3) The bandwidth parameter $q(t)$ is a function of only the damping ratio and the time normalized by the natural period T_0 of the SDOF system. In particular, the stationary value of $q(t)$ depends only on the damping ratio of the SDOF system. In fact, it can be shown from Eqs. (20) and (48) that

$$q_\infty = \lim_{t\to\infty} q(t) = \left\{1 - \frac{4\left[\operatorname{arctg}\left(\sqrt{1-\xi^2}/\xi\right)\right]^2}{\pi^2(1-\xi^2)}\right\}^{\frac{1}{2}}. \tag{51}$$

This stationary value decreases with decreasing value of ξ with $\lim_{\xi\to 0} q_\infty = 0$ indicating that the response process after reaching stationarity approaches a single harmonic component (with random phase and amplitude) as the damping ratio approaches zero.

Fig. 3 shows the ratio of the central frequency of the displacement response process over the natural circular frequency, referred to as the normalized central frequency, of SDOF systems with varying damping ratio ($\xi = 0.01, 0.05$ and 0.10). It is observed that:

(1) The normalized central frequency has a very high value at small (t/T_0), then as (t/T_0) increases it reaches a minimum and finally oscillates until it reaches stationarity. These oscillations remain always below the stationary value.
(2) The normalized central frequency is a function of only the damping ratio and the time normalized by the natural period T_0 of the SDOF system. The stationary value of the

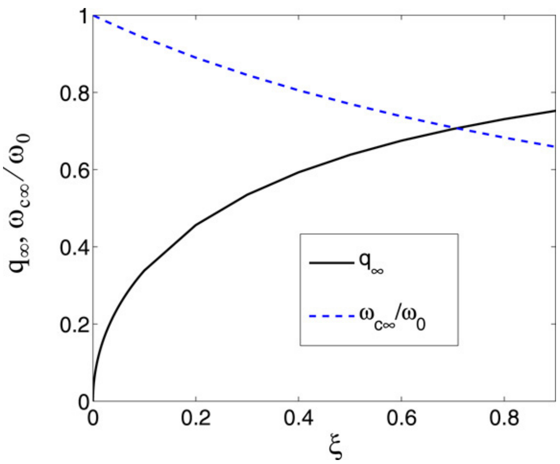

Fig. 4. Dependency on damping ratio of the stationary values of the bandwidth parameter and normalized central frequency, respectively, of the displacement response of a linear SDOF system.

normalized central frequency depends on the damping ratio only. This stationary value is given by [22]

$$\omega_{c\infty}/\omega_0 = \lim_{t\to\infty}[\omega_c(t)/\omega_0] = \frac{1}{\sqrt{1-\xi^2}}\left[1-\frac{2}{\pi}\text{arctg}\left(\xi/\sqrt{1-\xi^2}\right)\right]. \quad (52)$$

In particular, $\lim_{\xi\to 0}(\omega_{c\infty}/\omega_0) = 1$, which implies that the single harmonic component (with random phase and amplitude) approached by the displacement response process at large (t/T_0) for a lightly damped SDOF system has a frequency equal to the natural frequency of the system.

Fig. 4 shows the dependency of the stationary values of the bandwidth parameter and normalized central frequency, respectively, on the damping ratio for a linear SDOF system, summarizing in graphical form some of the above observations. Fig. 5 provides a single realization of a white noise excitation with PSD $\phi_0 = 0.01$ m^2/s^3 and the corresponding displacement response histories of linear SDOF systems with natural period $T_0 = 1.0$ s and damping ratio $\xi = 0.01, 0.05$ and 0.10. The displacement time-histories corresponding to $\xi = 0.01$, after a few seconds of transient behavior, clearly approach a single harmonic component with a mean frequency close to the natural frequency of the system, as indicated by the results shown in Fig. 2 for the bandwidth parameter and in Fig. 3 for the normalized central frequency. For the higher damping ratios of $\xi = 0.05$ and 0.10, after the initial transient behavior, a predominant harmonic component can also be observed in the displacement response histories. However, for these two higher damping cases and particularly for $\xi = 0.10$, contributions to the displacement response histories from other frequency components are non-negligible (broadening the frequency bandwidth of the response).

5.2. Three-story shear-type building (linear MDOF system)

The three-story one-bay steel shear-frame shown in Fig. 6 is considered as an application example. This building structure

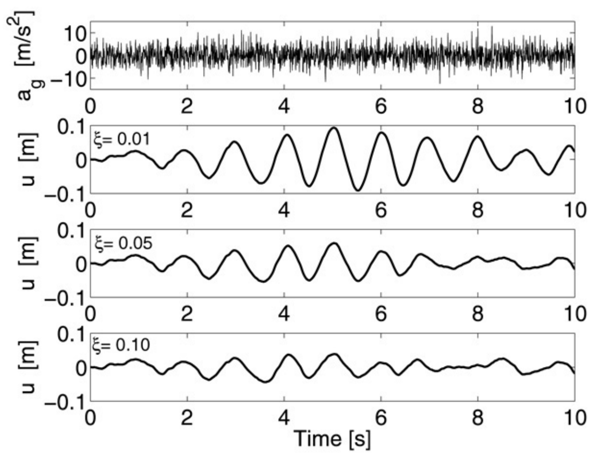

Fig. 5. Realization of white noise excitation ($\phi_0 = 0.01$ m^2/s^3, $\Delta t = 0.005$ s) and corresponding displacement response histories of linear SDOF systems with natural period $T_0 = 1.0$ s and damping ratios $\xi = 0.01, 0.05$ and 0.10.

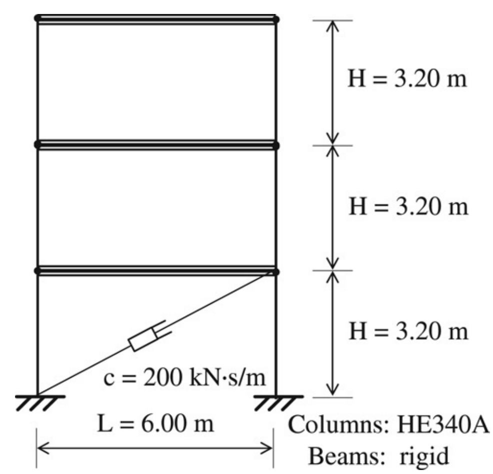

Fig. 6. Geometric configuration of benchmark three-story one-bay shear-type steel frame.

has a uniform story height $H = 3.20$ m and a bay width $L = 6.00$ m. The steel columns are made of European HE340A wide flange beams with moment of inertia along the strong axis $I = 27690.0$ cm^4. The steel material is modeled as linear elastic with Young's modulus $E = 200$ GPa. The beams are considered rigid to enforce a typical shear building behavior. Under this assumptions, the shear-frame is modeled as a 3 DOF linear system.

The frame described above is assumed to be part of a building structure with a distance between frames $L' = 6.00$ m. The tributary mass per story, M, is obtained assuming a distributed gravity load of $q = 8$ kN/ m^2, accounting for the structure's own weight, as well as for permanent and live loads, and is equal to $M = 28\,800$ kg. The modal periods of the linear elastic undamped shear-frame are $T_1 = 0.38$ s, $T_2 = 0.13$ s and $T_3 = 0.09$ s, with corresponding effective modal mass ratios of 91.41%, 7.49% and 1.10%, respectively. Viscous damping in the form of Rayleigh damping is assumed with a damping ratio $\xi = 0.02$ for the first and third modes of vibration. The same shear-frame is also considered with the addition of a viscous

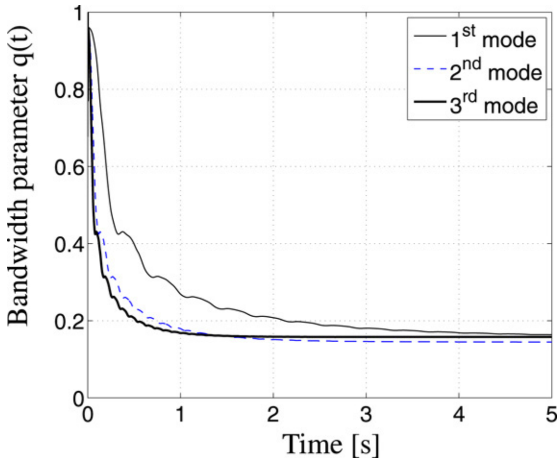

Fig. 7. Time-variant bandwidth parameter of each of the three modal displacement responses of the (classically damped) three-story shear-frame.

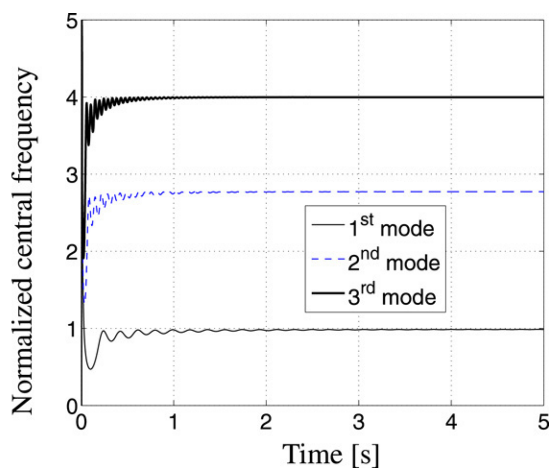

Fig. 8. Time-variant central frequency (normalized by first-mode natural frequency) of each of the three modal displacement responses of the (classically damped) three-story shear-frame.

damper of coefficient $c = 200$ kN s/m across the first story as shown in Fig. 6. The structure with viscous damper is a non-classically damped system. In both cases (with and without viscous damper), the shear-frame is subjected to base excitation modeled as a Gaussian white noise with PSD $\phi_0 = 0.1$ m^2/s^3 time-modulated by the unit-step function (i.e., from at rest initial conditions).

Figs. 7 and 8 show the bandwidth parameter and normalized central frequencies (central frequency divided by the natural circular frequency of the first mode of vibration), respectively, for each of the three modes of vibration of the shear-frame. The stationary values of the bandwidth parameters for the first and third modes are identical, since these two modes have the same damping ratio, see Eq. (51). The second mode has a lower damping ratio ($\xi_2 = 0.017$) and therefore a lower stationary value for the bandwidth parameter.

Figs. 9 through 11 show time histories of the variances of the floor displacements (relative to the ground), and of the bandwidth parameters and central frequencies (normalized by the first-mode natural frequency) of the floor relative displacement responses for the classically damped case. These figures show that the floor relative displacement response processes are dominated by the first-mode contribution. In

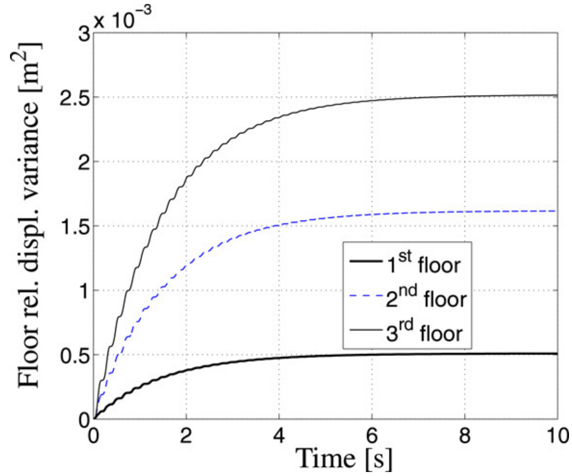

Fig. 9. Time-variant variances of floor relative displacement responses of three-story shear-frame without damper (i.e., classically damped).

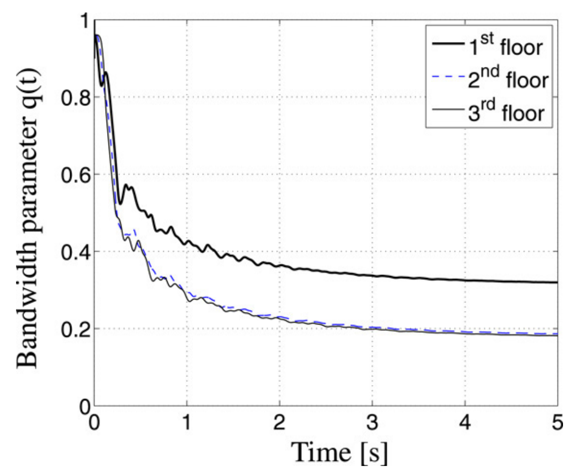

Fig. 10. Time-variant bandwidth parameters of floor relative displacement responses of three-story shear- frame without damper (i.e., classically damped).

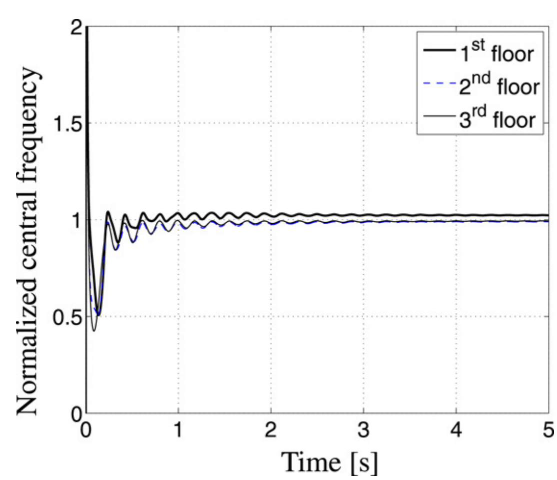

Fig. 11. Time-variant central frequencies (normalized by first-mode natural frequency) of floor relative displacement responses of three-story shear-frame without damper (i.e., classically damped).

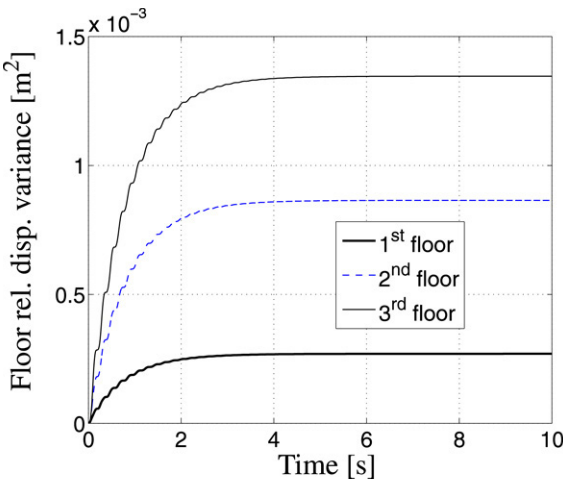

Fig. 12. Time-variant variances of floor relative displacement responses of three-story shear-frame with damper (i.e., non-classically damped).

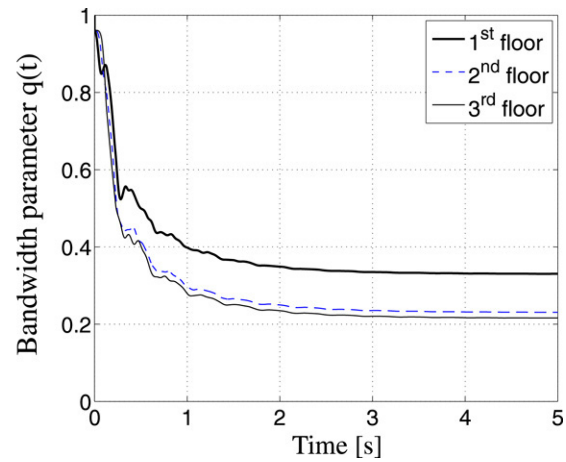

Fig. 13. Time-variant bandwidth parameters of floor relative displacement responses of three-story shear-frame with damper (i.e., non-classically damped).

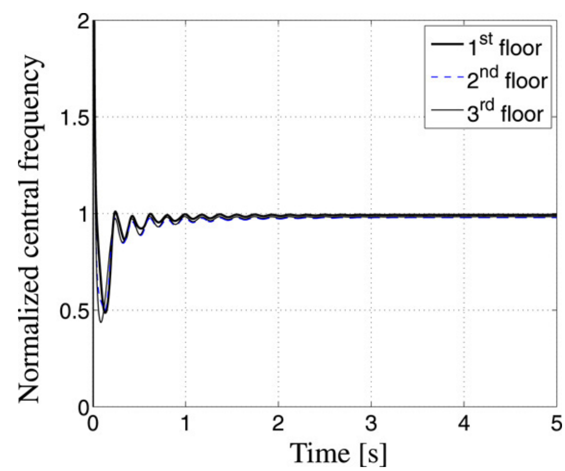

Fig. 14. Time-variant central frequencies (normalized by first-mode natural frequency) of floor relative displacement responses of three-story shear-frame with damper (i.e., non-classically damped).

particular, the time-histories of the bandwidth parameters and normalized central frequencies of the floor (especially the second and third floors) relative displacement responses are very similar to their counterparts for the first mode as shown by comparing Fig. 7 with Fig. 10 and Fig. 8 with Fig. 11, respectively. This comparison also indicates that the first floor relative displacement response has some small higher-mode contributions.

Figs. 12 through 14 provide the same information as Figs. 9 through 11, but for the shear-frame with viscous damper (i.e., non-classically damped case). The floor relative displacement response processes remain dominated by the first-mode contribution. The higher damping ($\xi_1 = 0.037, \xi_2 = 0.048, \xi_3 = 0.034$) reduces significantly the variances of floor relative displacements as shown by comparing Figs. 9 and 12. The higher damping has also the effect of raising slightly the stationary value of the bandwidth parameters of the floor relative horizontal displacements.

This second application example illustrates the capability of the presented extension of non-geometric spectral characteristics to complex-valued stochastic processes to capture the time-variant spectral properties in terms of the bandwidth parameter and central frequency of the response of linear MDOF classically and non-classically damped systems.

6. Conclusions

This paper extends the definition of the non-geometric spectral characteristics (NGSCs) to general complex-valued non-stationary random processes. These newly defined NGSCs are essential for computing the time-variant bandwidth parameter and central frequency of non-stationary response processes of linear systems. The bandwidth parameter is also used in structural reliability applications, e.g., for obtaining analytical approximations of the probability that a structural response process outcrosses a specified limit-state threshold.

Using the non-geometric spectral characteristics of complex-valued non-stationary processes and employing complex modal analysis, closed-form exact solutions are found for the classical problem of deriving the time-variant central frequency and bandwidth parameter of the response of linear SDOF and MDOF systems, both classically and non-classically damped, when subjected to white noise excitation from at rest initial conditions.

The exact closed-form solutions derived for the linear SDOF oscillator are used to investigate the dependency of the stationary and time-variant central frequency and bandwidth parameter on the SDOF system parameters, i.e., natural circular frequency, ω_0, and damping ratio, ξ. A three-story shear-type steel frame building without and with a viscous damper (i.e., classically and non-classically damped, respectively) is used to illustrate the application of the presented closed-form solutions for linear MDOF systems to the floor response processes of a base excited building structure.

The exact closed-form solutions developed and presented in this paper have their own mathematical merit, since to the best of the authors' knowledge, they are new. These solutions have a direct and important application, since the response of many structures can be approximated by using linear SDOF and MDOF models, and provide valuable benchmark solutions for validating (at the linear structural response level) numerical methods developed to estimate the probabilistic response of non-linear systems subjected to non-stationary excitations.

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